# Everything you need to know about integration

you already know the intuition behind the logic of integrals. Basically, it’s a bunch of rectangles that approximate the area under a curve, and as the number of rectangles approaches infinity, this approximation approaches the area under the curve. However, proving integrability puts you in the same spot as you were with proving differentiability[[1]](#footnote-1). The link between derivatives and integrals has not been made yet, and you’ll see why when you prove a function is integrable – it’s not intuitive when doing the proof of integrability that it could link to derivatives.

Mathematically, proving integrability in MAT137 in 2017-18 uses Darboux sums, which are more elegant version of Riemann sums. Since you already have an intuition, I’ll jump into the definition and sort things out from there

## Key ideas for MAT137

In MAT137, you need to know how to prove integrability. This means you need to translate your intuitions into math. The math of ever increasing approximations is represented via the epsilon-delta paradigm.

You know you need to take a partition of the area under the integral. Choosing the right partition is a very important step. A couple of hints would be to split functions into monotone sections (strictly increasing or decreasing) and try to take partitions of equivalent sizes. The number of partitions will always increase, precisely because you want to keep shrinking it to smaller rectangles for better approximations. This will be represented by “n”. You want to include “n” in the final equation, so that you could get every rectangle in the form of similar to:

So that you could do as in trying to prove the limit of horizontal asymptotes. If this doesn’t look familiar look at “everything you need to know about limits”.

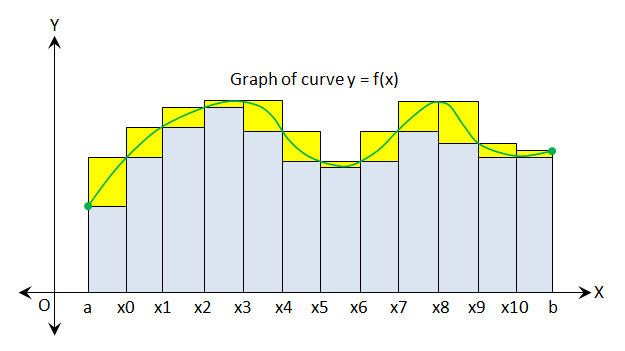
## Definition – without explanation

I’ll go strait to the epsilon-delta paradigm in which you’ll be doing most of the proving. Try seeing if you could compose this into something that makes sense before it’s explained:

The lower and upper Darboux sums of F for are:

A function is integrable if:

Visual: the blue is the lower Darboux sums – the yellow is what needs to be added for upper Darboux sums



## Explanation

* We’re only looking at a bounded part of the function. This method does not prove that a whole function is integrable (indefinite integral) or that the whole function integrates to a finite value (improper integral), but only part of the function is integrable (definite integral)
* We’re not integrating but proving integrability. However, there will be noticeable comparisons between the end result of the proof and regular integration. This will be explored in the “Fundamental theorem of Calculus”.

Partition:

* Definition: Given a finite interval [a,b], a partition of [a,b] is an ordered finite collection of points, including a,b
* Where:
* Note that sometimes this is denoted as , but this could be confused with the symbol for cardinality. does not have cardinality n
* This is key to Solving an integral, since taking ever finer partition (keep growing) get’s us closer to the area under the curve[[2]](#footnote-2). Notice also that every point in the partition is bigger than the former and smaller than the next. This the mathematical way of splitting the function without confusion.
* You could also “manually” get the partition finer. This is call refining.
  + Definition: If of [a,b] and , is a refinement of .
* Refinements are important for many proofs. Considering the upcoming Term test 3 will have at least 4 proof questions, and there are 6 questions total, this is important.
* In MAT157, we proved that a refined partition could be used in place of a partition. For MAT137, I believe this is self-evident (ask Tyler)

Darboux sums

* This is the proto-integral. As get’s larger, this get’s closer the integral.
* Note that sometimes the Darboux sum doesn’t exists. For example: on the interval is not integrable.
* Definition: Given a bounded f, we define the upper Darboux sum on a partition P as
* Equivalently, we define the lower Darboux sum as
* Note that these Darboux sums are linear. So sum(a+b) = sum a + sum b and sum(ca) = c\*sum(a).
* If we take a uniform partition, we could simplify the () to because:
* Another challenge will be to find the infimum and supremum of the Darboux sum. Some functions are easy, (like f(x) = x), but it might take a lot of work other times.

Darboux integrals:

* As n approaches , we get the [Darboux] integral.
* There is a lower and upper Darboux integral:
* Definition: the Lower Darboux integral of f on [a,b] is
  + The top of the integral should have a bar over it, not the b
* Equivalently for lower Darboux sum:
  + The bottom of the integral should have a bar over it, not the a
* Since the partition get’s finer, we could see that the upper limit the lower Darboux sum and the lower limit of the upper Darboux sum approach each other. Namely:
* This means that with an appropriate partition approaching infinite size, this should approach the area under the integral.
* The previous equation is actually another way to proof integrability. In assignment 8, we’ve proven how this is equivalent to the aforementioned definition.
  + You could put absolute value brackets around the U – L, but it doesn’t to anything so it’s redundant
* In MAT157, we define the properties of Darboux sums, but this is not necessary for MAT137.

Professor Holden made this analogy:

* Differentiable has the same use as integrable
* Derivative has the same use as integral

## Notation

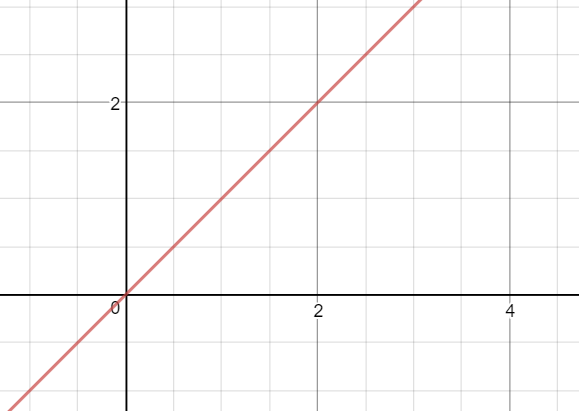
## behind **proving** integrability

As mentioned ad nauseum, the key is to find a good enough partition such that the upper and lower sums get epsilon close to each other. IT is also important to find the supremum and infimum of a function, which we’ll get into later. The partition will affect the upper and lower Darboux sums to create a difference between U and L smaller than . in the following example, I’ll show how finding the right partition is important:

1. Show that f(x) = x is integrable on

Proof:

Visually, this graph is



we do need to find a good partition. Your go-to will be a uniform partition. So, define

A couple of important notions to point out. The “n” represents the size of the partition, and it will get bigger. Visually, it will be analogous to:

<https://upload.wikimedia.org/wikipedia/commons/0/0a/Riemann_Integration_and_Darboux_Upper_SuEms.gif>

Since the partition is uniform, we could replace () as . Now we check to see how this partition influences the Darboux sums:

Where . The supremum could be difficult to determine in certain situation and would need a lot of thinking. In this case, it is equivalent to right Riemann sums.

The infimum for the lower sum is very much similar except we take the lower Darboux sum:

Now that we’ve established a formula for both sums. We could continue by trying to:

Note that starting at 0 and ending at (n-1) will give the same result.

Notice this turns out to be a telescoping sum. This will happen quite often as there’s a lot of overlap between upper and lower Darboux sums. One just needs a keen eye to look out for it:

Now replacing substituting back, the value of , and , we get:

We’re going to label this top equation (\*) to refer to it in our proof, since this is very important. Now we could pick an epsilon such that

If it’s unclear why you could just do this, look over “everything you need to know about limits”. Once you’ve gotten this far, it’s quite simple to proof:

* Let be given and choose sufficiently large such that
* Let P be a uniform partition[[3]](#footnote-3)
* By (\*)
* Q.E.D.

You could also take a different approach. Starting from

Note that

Then you could work from there. A lot of things will be grought out the the parenthesis and you’ll get the same result (since the inf(f(x\_i)) = x\_i)

You

1. Show that is integrable.

Proof:

Visually, the graph is



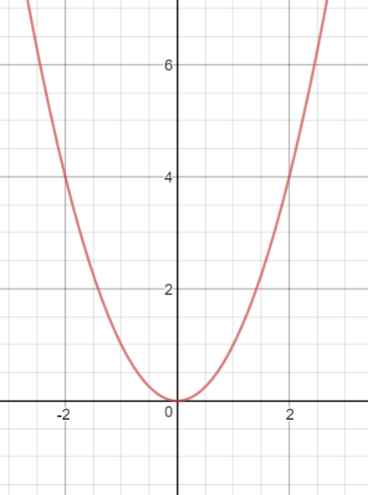
This means that choosing a partition here will be very easy. Since the supremum and infimum of F(x) for any value of f(x) will always be c(b-a). This means that we’re going to cheat a bit for this proof. We’re not going to choose 1 partition, but let P be any partition:

* Let be given
* Let P be any partition (this is clear in the next step)
* Since sup(f) = inf(f) for the partition
* Therefor

Look over this to see why this is a speial case.

1. Prove that f(x) = is integrable on

Visually



proof: This one is more complicated than the rest. The most important part is the intuition behind how to substitute the supremum value with a choice of 4 much easier variables to work with.

First. Let

Since the function is strictly increasing, the supremum and infimum will be the right and left reimnans sums respectively

Here’s the important insight. Working with is vey challenging and not practical to solve such summands. However, notice that it’s equivilant to

Since n represents the number of partitions, represents what point of the partitions we’re looking at and a, or more specifically (a-0) is scaling to the appropriate value, since the interval is . Therefore, we get

You need to memorize some standard summands. You could derive them as well, but some are simply useful to know:

Now set this value less than epsilon

From here, you could finish the proof.

### More information

One must be conscious of discontinuous function, since you could integrate a discontinuous function like

This function is integral, and the integral from 0 to 2 has the value of 5. Once again, the key is in the partition. This is officially called zero measure.

On the 2017-18 MAT137 test, you only really needed to know how to prove that x or is integrable.

You also might need to know a couple of basic summations like

This could be useful when you found the supremum of the function to be rearranged in a function of i, but don’t have time to solve for i. Here’s a shortcut to derivating these formulas.

1. Remember that 1+2+…+10 = 55. Remember noticing that (10+11)/2=55 and that the general pattern held. Mathematically, this is written as
2. To find the next one, we’re going to take advantage of binomal expansions and this formula
3. Start with
4. Rearrange so that the left-hand side becomes a telescoping series if summed[[4]](#footnote-4). Sum both sides
5. Replace the value for the sum of k, the telescoping series, and the sum of 1. Isolate the sum of k^3
6. This is finished, you could also try to rearrange to get a nice equation like so:
7. Now that you know k^2, you could move on to the next binomial expansion and you’d replace k^2 with this equation.

## properties of integrals

Now we’re going to bootstrap some properties. Like derivatives, integrals are linear:

### linear

Proof of (1)

Let

First, we must show that is integrable. This means we have to show that

As always, we need find to find a partition. We know the partition for f(x) and g(x) exist, since our goal is not to show that those function are integrable, but once we know they’re integrable, we can combine them via addition. Therefore let:

[[5]](#footnote-5)

Now here’s the trick: Darboux sums follow all the properties of summation, **which means we could separate them.** To find a new partition, we are given that the refinement of two partitions keep integrability. The refinement is done by taking their union, as explain in the ‘pertinent definition section’:

By the properties of summation:

Bringing back the equation we’re trying to solve, we could know plug in our new equation to get:

This proves that the equation is integrable. Now we need to prove that the integral of f+g is linear.

In this section, Holden skipped a lot of steps, so I’ll as him to make sure what I’m saying here is true (Tyler here). In here, a lot of the same technics of taking advantage of summation is linear will be used. The second definition of integrability is

Since we’ve proven the function is integrable, we only need to work with one side of the equation, so let’s work with the left side. What we want to prove is:

Which is mathematically equivalent to:

We’ll work on the supremum version. Looking at the definition of the supremum we get:

Once again, we’ll lean on the fact that this holds for f and g. Since f and g are integrable, we could write their supremum just like the previous equation

We could now refine: . By properties proved in MAT157, we know that

Combing the equations, we get;

This is actually the end of the proof. We could work backwards from here to show that

Proof of (2):

Starting from proving it’s integrable, we look at the definition:

Since these are Darboux sums, they follow the principles of summation:

### Decomposing the interval

Proof:

Since the functions are integrable let

Take as a partition of . With some properties of summation, it’s not to hard to show that . Now follows a chain of equalities

### Swapping the indices

Proof:

Using the previous proof, we know that

PROOVE THAT

Doing some rearranging, we get:

### Relation to inequality

We’ve also got a relation for inequalities

Proof: somewhere in homework

## Unproven theorems

There were two theorems shown in class but not proven

**Theorem**: IF f is bounded and monotone, it is integrable

**Theorem**: if f is continuous, then f is integrable

In 137, we don’t need to prove the first theorem, but the second one was an exercise in the book.

## Fundamental Theorem of Calculus (FTC)

As you know, integrals are anti-derivatives. You must know the proof between these. First, the two parts of the theorem:

Suppose is integrable

1. The function is continuous on . Moreover, anywhere f is continuous, is differentiable. In this case, F is an anti-derivative of “f”.

Intuition: This is the formal way of getting an “anti-derivative” function.

1. iF F is a continuous anti-derivative for f which is differentiable at all but finitely many points (me: the function has zero-measure), then

Intuition: This is how we calculate precise values for functions

### Proof of (1):

Let’s show that F is continuous on [a, b] and differentiable on [a,b]. Starting with continuity:

In EYNTKA limits, it is briefly mentioned one way to proof continuity. However, Tyler uses a different form. This very much looks like uniform continuity[[6]](#footnote-6) without the |x-y| < delta (ask Tyler).

It suffices to show that

This is a definition of [uniform?] continuity. To prove the above inequality, we need to start by

The last part of that statement makes things easier and doesn’t loose generality. Now we want to introduce F(y) and F(x). We could do this by using a simple equality and a property of integrals:

The left integral on the L.H.S is equivalent to F(x), R.H.S to F(y). Doing the substitution and some rearranging, we get

The R.H.S is a nuisance. I’d recommend some pondering before I reveal the next step. I assume you’ve done so here we go. Notice that f is bounded on [a,b]. Therefore

From there, define epsilon to be

Now we need to prove differentiability. Looking at EYNTKA derivatives, recall that

Therefore, we want to show that

There’s going to be a long chain of equalities and inequalities. Look at them carefully. First, take advantage of the fact that f is continuous, so:

Let be given. Since f is continuous at x, s.t.

Q.E.D.[[7]](#footnote-8)

### Proof of (2)

HERE

### Proof of properties

A lot of the properties were proven in MAT157, however one is needed for MAT137.

is equivalent to

Derivative of this equation is insightful. On the right-hand side, the left element is actually a constant, therefore it disappears. The right equation is back to the original equation, therefore the value of a is not important.

## Indefinite integrals

We could now integrate functions. The notation is

Remember: do NOT forget the **+ C**

For this, you simply need to remember a lot of standard “anti-derivatives” and two other facts that might trick you on a test or exam:

Left hand side

Right and side:

You also need to know how the integral signed is used differently, and what it represents

The first integral represents a number. The second integral represents a function. The third one represents a set of functions. You could imagine that the middle is the subset of the latter.

## Other types of integrals

These types of integrals will be relegated to “EYNTKA integration application”, mainly because the computation of those integrals starts getting into the realm of using integrals versus proving integrability. These integrals include

* improper integrals
* integration of solids using revolution technics (Washer method, disk method, etc.)

## Insight into hard integrals

In this section, I’ll run over some hard integrals to give an insight into problem solving technics.

1. **DON’T FORGET C!!**
2. Proving integrability of certain function could be hard. However, you could quickly enough prove that for any bounded increasing function:

For a uniform partition, it get’s even nicer:

1. If you’re looking at a problem and your trying to think of scenarios, and one of the scenarios you keep thinking up is getting smaller and smaller values, consider using epsilon and deltas. For example, to prove

You started by assuming that and took for simplicity. You wanted to create an integral with this function, but you kept shrinking it and couldn’t be sure that the neighbourhood of f(c) is also increasing. However, since the function is continuous, **you could set s.t. f(x) >0 for all**

This is incredibly useful, since now you could do

Then proving that our first integral is not equal to zero is by simply splitting it

## Proving a function is not integrable

We were not taught this in class, therefore this will be a filler section in case it will be introduced another year.

1. Though now you know a bit more on this new paradigm, lucky you 😊 [↑](#footnote-ref-1)
2. Online, I’ve found that this is called Lebesgue integrability . [↑](#footnote-ref-2)
3. This is a shortcut the partition we wrote earlier, and we’re allowed to say this on Tyler’s 2017-18 term test and exam. [↑](#footnote-ref-3)
4. A teslescoping series is one were all but the first and last elements cancel [↑](#footnote-ref-4)
5. We’ve proved that it is equivalent to choose vs . [↑](#footnote-ref-5)
6. is it Lipschitz. [↑](#footnote-ref-6)
7. One step is not explained: . I don’t know if we need to know this (proof in “calculus in one-dimension volume 2” p.477). ask Tyler [↑](#footnote-ref-8)